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1993 J. Phys. A: Math. Gen. 26 L661

(http://iopscience.iop.org/0305-4470/26/15/008)

J. Phys. A: Math. Gen. 26 (1993) L661–L665. Printed in the UK

LETTER TO THE EDITOR

Deformation of the strange superalgebra $\tilde{P}(n)$

L Frappat†§, P Sorba†§ and A Sciarrino‡||

† Laboratoire de Physique Théorique LAPP, BP 110, F-7494 Annecy-le-Vieux Cedex, France and ENS Lyon, 46 Allée d'Italie, F-69364 Lyon Cedex 07, France
‡ Università di Napoli Federico II, Mostra d'Oltremare Pad. 19, I-80125 Napoli, Italy and I.N.F.N., Sezione di Napoli, Italy

Received 20 November 1992

Abstract. A deformation $U_q(\tilde{P}(n)) = \tilde{P}_q(n)$ of the extended non-contragredient (strange) superalgebra of $P(n)$, denoted by $\tilde{P}(n)$, is proposed. A realization of $\tilde{P}_q(n)$ in terms of q-bosons and q-fermions is presented. Then a procedure to build up a set of representations of $\tilde{P}_q(n)$ is briefly discussed. As a by-product, a new realization of $SU_q(n)$ is obtained.

The quantum groups are today a topic of intensive research both in mathematics and physics. The q-deformed universal enveloping algebra $U_q(\mathscr{G})$ or $\mathscr{G}_q$ of a semi-simple Lie algegra $\mathscr{G}$ is defined by a set of q-depending relations between the generators of $\mathscr{G}$ in the Serre presentation endowed with a Hopf algebra structure, using as input the Cartan matrix of $\mathscr{G}$ (see for instance [1] for more details). It is a natural idea to apply these tools to the Lie superalgebras (or SLA's) using as input the defining Cartan matrix of SLA's [2]. Introducing a set of q-deformed bosons and fermions, explicit realizations of classical SLA's [3] as well as exceptional SLAs [4] have been obtained. We will show in this letter that the q-deformation procedure can also be applied to the strange non-contragredient superalgebra $P(n)$ [5], or more precisely to its extension $\tilde{P}(n)$. As presented in [6], a non-degenerate Killing form and a generalized rectangular Cartan matrix can be constructed for this non-contragredient Lie superalgebra. One will see that such a Cartan matrix is well adapted to get the Serre relations for $\tilde{P}(n)$ in the quantum case, and also that a Hopf structure can be proposed. Moreover, the oscillator realization introduced in [7] can be deformed, in a rather non-trivial way, to produce a realization of $\tilde{P}(n)$.

Let us recall that the superalgebra $\tilde{P}(n)$ is generated in the distinguished basis by $3(n-1)$ bosonic generators of $SU(n)$, E_i, F_i, H_i with $i = 1, \ldots, n-1$, a fermionic

e/mail: FRAPPAT@FRLAPP51 or SORBA@FRLAPP51
§ URA 14-36 du CNRS, associée à l'E.N.S. de Lyon et au L.A.P.P. d'Annecy-le-Vieux (IN2P3-CNRS)
Work supported in part by contract CEE SC 1000221 and by Ministère des Affaires Etrangères in France.
|| e-mail: SCIARRINO@NA.INFN.IT

generator E_n and a diagonal generator D such that (we refer to [6] for more details, in particular for the study of the root system)

$$\left.\begin{aligned}
&[H_i, H_j]=0\\
&[H_i, E_j]=a_{ij}E_j\\
&[H_i, F_j]=-a_{ij}F_j\\
&[E_i, F_j]=\delta_{ij}H_i
\end{aligned}\right\} \quad \text{for } 1\leqslant i,j\leqslant n-1 \tag{1a}$$
$$[H_i, E_n]=a_{in}E_n$$

and

$$[D, E_i]=[D, F_i]=[D, H_i]=0 \quad \text{and} \quad [D, E_n]=E_n \tag{1b}$$

where $(a_{ij})_{1\leqslant i,j\leqslant n-1}$ is the Cartan matrix of $SU(n)$ and $a_{in}=0$ for $1\leqslant i\leqslant n-2$ and $a_{n-1,n}=-2$.

One can arrange the coefficients a_{ik} ($1\leqslant i\leqslant n-1$ and $1\leqslant k\leqslant n$) in order to build the Cartan matrix of $\tilde{P}(n)$ such that

$$(a_{ik})_{\tilde{P}(n)}= \quad (a_{ij})_{SU(n)} \begin{pmatrix} 0\\ \vdots\\ 0\\ -2 \end{pmatrix}. \tag{2}$$

Moreover, one has the following Serre relations with $1\leqslant i,j\leqslant n-1$ and $1\leqslant k\leqslant n$:

$$\begin{aligned}
&(\mathrm{ad}\, E_i)^{1-a_{ik}}E_k=\underbrace{[E_i,\ldots[E_i}_{1-a_{ik}\ \text{commutators}}, E_k]\ldots]=0\\
&(\mathrm{ad}\, F_i)^{1-a_{ij}}F_j=[\underbrace{F_i,\ldots[F_i}_{1-a_{ij}\ \text{commutators}}, F_j]\ldots]=0.
\end{aligned} \tag{3}$$

Now we propose a deformation of the strange superalgebra $\tilde{P}(n)$ based on the following structure. Let us first define

$$[X]_q\equiv\frac{q^X-q^{-X}}{q-q^{-1}}. \tag{4}$$

Then the deformed superalgebra is given by

$$\left.\begin{aligned}
&[H_i, H_j]=0\\
&[H_i, E_j]=a_{ij}E_j\\
&[H_i, F_j]=-a_{ij}F_j\\
&[E_i, F_j]=\delta_{ij}[H_i]_q
\end{aligned}\right\} \quad \text{for } 1\leqslant i,j\leqslant n-1 \tag{5}$$
$$[H_i, E_n]=a_{in}E_n$$

and the quantum Serre relations read with $1\leqslant i, j\leqslant n-1$ and $1\leqslant k\leqslant n$:

$$\begin{aligned}
&\sum_{0\leqslant m\leqslant 1-a_{ik}}(-1)^m\begin{bmatrix}1-a_{ik}\\ m\end{bmatrix}_q (E_i)^{1-a_{ik}-m}E_k(E_i)^m=0\\
&\sum_{0\leqslant m\leqslant 1-a_{ij}}(-1)^m\begin{bmatrix}1-a_{ij}\\ m\end{bmatrix}_q (F_i)^{1-a_{ij}-m}F_j(F_i)^m=0.
\end{aligned} \tag{6}$$

Moreover, we impose the standard Hopf structure on the generators $E_i, F_i, H_i; D$ (for $i=1,\ldots,n-1$), with a comultiplication Δ, a co-unit ε and an antipode S such that

$$\begin{aligned}
&\Delta(E_i)=E_i\otimes q^{-H_i/2}+q^{H_i/2}\otimes E_i\\
&\Delta(F_i)=F_i\otimes q^{-H_i/2}+q^{H_i/2}\otimes F_i\\
&\Delta(H_i)=H_i\otimes 1+1\otimes H_i\\
&\Delta(D)=D\otimes 1+1\otimes D
\end{aligned}\tag{7}$$

and

$$\begin{aligned}
&\varepsilon(H_i)=\varepsilon(E_i)=\varepsilon(F_i)=\varepsilon(D)=0 \qquad \varepsilon(1)=1\\
&S(H_i)=-H_i \qquad S(D)=-D\\
&S(E_i)=-q^{-1}E_i \qquad S(F_i)=-qF_i.
\end{aligned}\tag{8}$$

Now we face some ambiguity in the definition of the coproduct of E_n, as this generator has no counterpart F_n. The consistency condition between the coproduct $\Delta(E_n)$ and the commutation relations $[H_i, E_n]=a_{in}E_n$ requires the coproduct to have the form

$$\Delta(E_n)=E_n\otimes q^{X/2}+q^{Y/2}\otimes E_n \tag{9}$$

where X and Y belong to the Cartan subalgebra of $\tilde{P}(n)$.

In order to solve this problem, let us remark that, in the case of contragredient superalgebras, there appears in the coproduct of a generator e_i, the operator h_i such that $[h_i, e_j]=(\alpha_i, \alpha_j)e_j$, α_i and α_j being the roots associated to the generators e_i and e_j respectively. In the case of $\tilde{P}(n)$, in [6], simple roots have been introduced such that

$$(\alpha_i,\alpha_i)=(\gamma_n,\gamma_n)=2 \qquad (\alpha_i,\alpha_j)=-\delta_{i,j-1}(i<j) \qquad (\alpha_i,\gamma_n)=-2\delta_{i,n-1} \tag{10}$$

where α_i are associated to the generators E_i and γ_n to the generator E_n.

Then we make the choice

$$X=-Y=\frac{2}{n}\left((2-n)D-\sum_{i=1}^{n-1} iH_i\right). \tag{11}$$

Finally, the co-unit and the antipode for E_n are chosen as

$$\varepsilon(E_n)=0 \qquad \text{and} \qquad S(E_n)=-q^{-1}E_n. \tag{12}$$

In the following, we give an explicit realization of the generators of $\tilde{P}_q(n)$ in terms of q-bosons b_i and b_i^+ and q-fermions a_i and a_i^+ with $1\leqslant i\leqslant n$, which is a natural deformation of the realization of $\tilde{P}(n)$ in terms of bilinear products of oscillators of bosonic and fermionic type given in [7].

In order to fix the notations, let us recall the definition of q-bosons:

$$\begin{aligned} & b_i b_j^+ - q^{\delta_{ij}} b_j^+ b_i = \delta_{ij} q^{-N_i} \\ & [N_i, b_j] = -\delta_{ij} b_j \\ & [N_i, b_j^+] = \delta_{ij} b_j^+ \end{aligned} \tag{13}$$

together with

$$\left.\begin{aligned} & [N_i, N_j] = 0 \\ & [b_i, b_j] = [b_i^+, b_j^+] = 0 \end{aligned}\right\} \qquad \text{for } i \neq j \tag{14}$$

and also of q-fermions:

$$\begin{aligned} & a_i a_j^+ + q^{\delta_{ij}} a_j^+ a_i = \delta_{ij} q^{M_i} \\ & [M_i, a_j] = -\delta_{ij} a_j \\ & [M_i, a_j^+] = \delta_{ij} a_j^+ \end{aligned} \tag{15}$$

together with

$$\left.\begin{aligned} & [M_i, M_j] = 0 \\ & \{a_i, a_j\} = \{a_i^+, a_j^+\} = 0 \end{aligned}\right\} \qquad \text{for } i \neq j \tag{16}$$

two sets of oscillators being commuting each other.

Moreover, one imposes the equations (13) to (16) to be invariant in the change $q \to q^{-1}$.

Then the realization of the generators of $\tilde{P}_q(n)$ in terms of the deformed oscillators is given by:

$$\begin{aligned} & E_i = b_i^+ b_{i+1} q^{(M_i - M_{i+1})/2} + a_i^+ a_{i+1} q^{-(N_i - N_{i+1})/2} \\ & E_n = b_n^+ a_n^+ q^{(\Sigma_{i=1}^{n-1} N_i - \Sigma_{i=1}^{n-1} M_i)/2} \\ & F_i = b_{i+1}^+ b_i q^{(M_i - M_{i+1})/2} + a_{i+1}^+ a_i q^{-(N_i - N_{i+1})/2} \\ & H_i = N_i - N_{i+1} + M_i - M_{i+1} \\ & D = \frac{1}{2} \sum_{i=1}^{n} N_i + \frac{1}{2} \sum_{i=1}^{n} M_i . \end{aligned} \tag{17}$$

Let us make a few remarks on the above equations:

(i) The realization is not invariant (except for the elements of the commuting subalgebra) in the change $q \to q^{-1}$.

(ii) For $q \to 1$, it reduces to the expression of the generators of $\tilde{P}_q(n)$ given in [7].

(iii) In the choice of E_n, it remains an ambiguity, for example one could multiply the expression of E_n by any factor q^{cD} where c is a c-number. As for the definition of the coproduct, the origin of the ambiguity is the non-existence of a conjugate operator F_n.

It is easy to verify that the above construction satisfies the defining equations (5). To show that the Serre relations are satisfied is a little more tricky but straightforward. It is useful to remark that for $a_{ij} = -1$, the Serre relation can be written as

$$[E_i, [E_i, E_j]_{q^{-1}}]_q = 0 \tag{18}$$

where $[E_i, E_j]_q = E_i E_j - q E_j E_i$, and for $a_{n-1,m} = -2$

$$E_{n-1}^2 [E_{n-1}, E_n] + [E_{n-1}, E_n] E_{n-1}^2 - (q^2 + q^{-2}) E_{n-1} [E_{n-1}, E_n] E_{n-1} = 0. \tag{19}$$

We may now consider representations of $\tilde{P}_q(n)$. Having used sets of q-bosonic and q-fermionic oscillators to realize $\tilde{P}_q(n)$, it is natural to try to build up representations of the deformed superalgebra on the Fock space of the oscillators, in analogy with the non-deformed case [6, 7]. To our knowledge, no general theorem exists for the representations of deformed Lie superalgebras. Moreover, even in the non-deformed case, no general theorem exists for the representations of $P(n)$. In [6], a heuristic procedure to build up a class of highest weight representations of $P(n)$ was presented and in [7] some representations were explicitly constructed in the Fock space of (non-deformed) bosonic and fermionic oscillators. Let us, along the same lines, define a highest weight representation of $\tilde{P}_q(n)$:

(i) There exists a (unique) highest weight vector $|\Lambda\rangle$ annihilated by the positive generators of $SU_q(n)$ and by E_n which are eigenvectors of the commutative subalgebra (H_i, D).

(ii) It decomposes into a sum of irreducible representations of $SU_q(n)$.

(iii) Any two vectors of the carrier space of the commutative subalgebra (H_i, D) are connected by raising and lowering operators.

Let us define the vacuum state $|0\rangle$ by

$$a_i|0\rangle = 0 \qquad b_i|0\rangle = 0 \qquad \text{with } 1 \leqslant i \leqslant n. \tag{20}$$

The highest weight vector must have the following form

$$|\Lambda\rangle = (b_i^+)^N \prod_{j=1}^{K} a_j^+ |0\rangle \tag{21}$$

where N and K are non-negative integers, such that $1 \leqslant K \leqslant n$.

The structure of the representations can be obtained as in [7], replacing the action of the fermionic and bosonic operators by the action of the corresponding q-deformed operators. In particular, one finds that the dimensions of the highest weight irreducible representations one can construct are the same for the deformed and the non-deformed superalgebras.

In conclusion, we have shown that the formalism introduced in [6], and in particular the use of a rather unconventional (rectangular) Cartan matrix, allow us to define for $\tilde{P}_q(n)$ a deformed superalgebra along the same lines as for classical Lie superalgebras. The existence in $\tilde{P}_q(n)$ of generators associated to roots which do not have the corresponding negative ones, leads to some subtitles in the deformation.

As a by-product of our realization, we have got a new realization of $SU_q(n)$, generated by E_i, F_i, H_i with $1 \leqslant i \leqslant n-1$ in terms of bilinears in q-bosons and q-fermions. The advantage of this constuction is to allow us to build up in the Fock space irreducible representations of $SU_q(n)$ corresponding to highest weight irreducible representations of $SU(n)$ labelled by Young tableaux of the type $[N+1, 1^{K-1}]$.

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